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Unit 9:
Lesson/ Module: **Electromagnetic Waves - III**

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Learning Objectives:

We continue to study propagation of electromagnetic waves. From this module students may get to know about the following:

1. *The propagation of electromagnetic waves in a dielectric medium. In this case the velocity of propagation depends on the frequency which leads to the phenomenon of dispersion.*
2. *The students are introduced to a simple model of a dielectric.*
3. *The phenomena of anomalous dispersion and the low and high frequency behaviour of the frequency dependence obtained.*
4. *Propagation of waves in a conducting mediumis described..*

Electromagnetic Waves – III
Wave propagation in dielectrics and conductors

9.1 Wave Propagation in Dielectrics

Electromagnetic waves propagate in a medium with a velocity $v = 1/\sqrt{\varepsilon\mu}$, where ε is the permittivity of the medium and μ is the permeability. In general both these quantities depend on the frequency of the propagating electromagnetic wave. Only in the case of vacuum is the velocity independent of the frequency. For dielectrics, the velocity can be treated as independent only over a limited range of frequencies. The frequency dependence of the velocity of propagation leads to the phenomenon of dispersion, spreading of waves and group velocity as distinct from the phase velocity. In this module we wish to study a simple model of dispersion for the purpose of studying these phenomena.

9.1.1 A simple Model

Consider an electron of mass *m* and charge *e* bound in the dielectric by a harmonic force

$$\vec{F} = -m\omega_0^{\ 2}\vec{x} \tag{1}$$

Here $\vec{x}$ is the displacement of the charge and ω_0 the frequency of oscillation about the equilibrium. If there is also present a damping force $m\gamma\vec{x}_0$, then when the charge is placed in an external field $\vec{E}(\vec{x},t)$, its equation of motion is

$$m[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^{\ 2}\vec{x}] = e\vec{E}(\vec{x},t) \tag{2}$$

In non-magnetic materials the effect of magnetic field is usually small and has been neglected here. If the intensity of the electromagnetic wave impinging on the medium is not very high, the amplitude of oscillation is small and we can evaluate the field on the right hand side at the position of equilibrium of the charge. Let it be $\vec{E}(t)$. If the field varies harmonically in time with angular frequency ω, then $\vec{E}(t) = \vec{E}e^{-i\omega t}$. In that case the solution can be written in the form

$$\vec{x}(t) = \vec{x}e^{-i\omega t} \tag{3}$$

On substituting into equation (2) we obtain the solution

$$\vec{x} = \frac{e}{m}\frac{1}{\omega_0^{\ 2} - \omega^2 - i\omega\gamma}\vec{E} \tag{4}$$

The dipole moment of the charge is given by

$$\vec{p} = e\vec{x} = \frac{e^2}{m}\frac{1}{\omega_0^{\ 2} - \omega^2 - i\omega\gamma}\vec{E} \tag{5}$$

Let there are be N molecules of the dielectric per unit volume and Z electrons per molecule. Further all the electrons in the molecule may not have the same binding frequency and damping constant. Let there be f_j electrons per molecule with binding frequency ω_j and damping constant γ_j Then

$$\sum_j f_j = Z \tag{6}$$

The dielectric constant κ is then given by

$$\kappa(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\omega_j^{\,2} - \omega^2 - i\omega\gamma_j} \tag{7}$$

For a proper understanding of ω_j, γ_j and f_j, we have to appeal to quantum mechanics. With suitable quantum-mechanical definition of these quantities, equation (7) provides a reasonably good description of the atomic contribution to the dielectric constant, κ.

9.1.2 Anomalous Dispersion

The damping constants γ_j are usually small compared to the resonant frequencies ω_j. This implies, from the above equation (7) that the dielectric constant κ has only a small imaginary part for most frequencies. The factor $\omega_j^{\,2} - \omega^2$ is positive for $\omega < \omega_j$ and negative for $\omega > \omega_j$ .As a result for "low" frequencies, low meaning frequency less than the lowest resonant frequency, all the terms contribute with the same positive sign and the dielectric constant $\kappa(\omega)$ is greater than unity. As the frequency is increased, successive resonant frequencies are passed, more and more terms contribute with negative sign until the whole sum becomes negative and the dielectric constant $\kappa(\omega)$ becomes less than unity. At any of the resonant frequencies, the real part vanishes and the corresponding term becomes pure imaginary and large, since γ_j is small. **[See Figure 7.8 frpm Jackson Ed 2]**

Away from the resonant frequencies, the dispersion is *normal dispersion* and close to the resonant frequencies it is *anomalous dispersion*. Except close to the resonant frequency, i.e.; in normal dispersion, the real part of the dielectric constant of the medium increases with frequency; the imaginary part remains small, usually negligibly small. On the other hand, in anomalous dispersion the real part drops and the imaginary part increases to a significant value.

It is also clear from equation (7) that the imaginary part is positive and a positive imaginary part implies dissipation of energy, i.e., transfer of energy from the electromagnetic wave to the medium, usually in the form of heat. The regions where *imaginary* part of κ is large are called regions of *resonant absorption*.

The attenuation of a wave is most easily expressed in terms of the wave number *k*. If the wave number is written as

$$k = k_{re} + i\frac{\alpha}{2} \tag{8}$$

then the parameter α is known as the *attenuation constant* or *absorption coefficient.* The amplitude of the wave falls off as $e^{-\alpha z/2}$ and hence the intensity as $e^{-\alpha z}$. Now the relation between the wave number and the dielectric constant is given by

$$k = \frac{\omega}{v} = \sqrt{\mu\varepsilon}\,\omega = \sqrt{\kappa\kappa_m\mu_0\varepsilon_0}\,\omega = \sqrt{\kappa\kappa_m}\,\frac{\omega}{c} \tag{9}$$

For non-magnetic materials, κ_m, the relative permeability, can be taken to be unity. On squaring and comparing real and imaginary parts, we have

$$k_{re}{}^2 - \frac{1}{4}\alpha^2 = \frac{\omega^2}{c^2}\mathrm{Re}(\kappa) \tag{10}$$

$$k_{re}\alpha = \frac{\omega^2}{c^2}\mathrm{Im}(\kappa) \tag{11}$$

Away from resonant absorption, $k_{im} << k_{re}$, and we can neglect the second term on the left hand side of equation (10). Hence

$$k_{re} = \frac{\omega}{c}\sqrt{\mathrm{Re}\,\kappa(\omega)} \tag{12}$$

$$\alpha = \frac{\mathrm{Im}\,\kappa(\omega)}{\mathrm{Re}\,\kappa(\omega)}k_{re} \tag{13}$$

The fractional decrease of intensity per wavelength is

$$1 - e^{-k_{im}\lambda} \approx k_{im}\lambda \approx k_{im}\frac{2\pi}{k_{re}} = 2\pi\frac{\mathrm{Im}\,\kappa(\omega)}{\mathrm{Re}\,\kappa(\omega)}$$

9.1.3 The Low Frequency Behaviour – Electrical Conductivity

Let us look at the low frequency limit of the behaviour first. The low frequency response depends crucially on whether the lowest resonant frequency is zero or nonzero. For insulators the lowest resonant frequency is different from zero. In that case, from equation (7) we obtain for the dielectric constant the expression (on putting γ_j =0)

$$\kappa(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\omega_j^{\ 2}} \tag{14}$$

which agrees with the expression familiar from electrostatics.

On the other hand if some fraction f_0 of the electrons are free in the sense that they have zero resonant frequency, the dielectric constant is singular at $\omega = 0$. If the contribution of the free electrons is shown separately, equation (7) for the dielectric constant takes the form

$$\kappa(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_{rest} \frac{f_j}{\omega_j^{\ 2} - \omega^2 - i\omega\gamma_j} + \frac{Ne^2}{\varepsilon_0 m} \frac{f_0}{\omega^2 + i\omega\gamma_0}$$

or

$$\kappa(\omega) = \kappa_0 + i \frac{Ne^2}{\varepsilon_0 m\omega} \frac{f_0}{\gamma_0 - i\omega} \tag{15}$$

where

$$\kappa_0 = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_{rest} \frac{f_j}{\omega_j^{\ 2} - \omega^2 - i\omega\gamma_j}$$

is the contribution of all the rest of the frequencies.

To understand the singular behaviour let us look at the Maxwell's equation

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Let us assume that the medium obeys Ohm's law, $\vec{J} = \sigma \vec{E}$ and has the normal dielectric constant κ_0. With harmonic time dependence we have $\frac{\partial}{\partial t} = -i\omega$, so that

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma\vec{E} - i\omega\varepsilon\vec{E} = -i\omega(\kappa\varepsilon_0 + \frac{i\sigma}{\omega})\vec{E}$$

If we do not insert Ohm's law explicitly but attribute all the properties of the medium to the dielectric constant, then on comparing with equation (15) we obtain

$$\sigma = \frac{Ne^2}{m} \frac{f_0}{\gamma_0 - i\omega} \tag{16}$$

This is basically the classical *Drude model* of conductivity. The damping constant γ_0 / f_0 can be determined empirically from the experimental data on conductivity. The quantity $f_0 N$ is the number of free electrons per unit volume in the medium. Putting in the experimental values of various quantities, we get $\gamma_0 / f_0 \approx 3\times 10^{13}\,\text{sec}^{-1}$. Assuming $f_0 \approx 1$, we see that for frequencies up to $\approx 10^{13}$ electrical conductivity of metals is essentially real and independent of the frequency. This range is well beyond the microwave region (ω up to around 10^{11}). In the higher frequency range of frequency the conductivity is complex, both real and imaginary parts are significant and it varies with frequency according to equation (16).

- Though classical theory gives an overall picture of conductivity, the problem basically is one of quantum mechanics. The concepts of Pauli exclusion principles, band theory of solids, free and valence electrons, all quantum-mechanical concepts play an important role.

- The distinction between conductors and insulators is an artificial one at least away from $\omega = 0$. If the medium possesses free electrons it is a conductor, otherwise an insulator. But at nonzero frequencies the contribution of conductivity to the dielectric constant is just like any other contribution from the resonant amplitudes. Whether we say that the dispersive properties of a medium are due to complex dielectric constant or due to frequency dependent conductivity are two different ways of saying the same thing.

9.1.4 High-Frequency Limit – Plasma Frequency

Now let us look at the behaviour of the medium at high frequencies; by high we mean that the frequency is well above the highest resonant frequency of the material. At such high frequencies equation (7) reduces to

$$\kappa(\omega) = 1 - \frac{{\omega_P}^2}{\omega^2} \tag{17}$$

where

$$\omega_P = \frac{NZe^2}{\varepsilon_0 m} \tag{18}$$

The frequency ω_P, which depends on the total number $N\sum f_i = NZ$ of electrons per unit volume, is called *plasma frequency* of the medium. In this limit the wave number is given by

$$k = \frac{\omega}{v} = \frac{\omega}{c}\sqrt{\kappa(\omega)} = \frac{1}{c}\sqrt{\omega^2 - {\omega_P}^2} \tag{19}$$

or

$$\omega(k)^2 = {\omega_P}^2 + c^2 k^2 \tag{20}$$

A relation like this between the frequency and wave number is called a *dispersion relation*. The dispersion relations and the analyticity properties of $\omega(k)$ in the complex *k*-plane play a very important role in quantum electrodynamics, particle physics and solid state physics etc.

In dielectrics equation (17) applies for $\omega >> \omega_P$. The dielectric constant is then close to but less than unity. It increases slightly with the frequency and approaches unity when frequency tends to infinity.

In certain situations, however, such as ionosphere the electrons are free and damping is negligible. Then equation (17) holds over a wide range of frequencies, including $\omega < \omega_P$. For frequencies less than the plasma frequency, the wave number is purely imaginary as can be seen from equation (19). Such waves, incident on a plasma are simply reflected (like total internal reflection) and the fields inside the plasma fall off exponentially with distance from the interface. The attenuation constant increases with decreasing frequency and approaches $2\omega_P / c$ in the small frequency limit.

On the laboratory scale, plasma densities are of the order of (10^{18} to 10^{22}) m^{-3}. From equation (18) this means ω_P is of order (6.10^{10}- 6.10^{12}) s^{-1}. The attenuation length which is inverse of attenuation constant lies in the range of 2 mm to 0.02 mm for low frequencies. The expulsion of fields from within a plasma is well known and is made use of in confinement of plasma in thermonuclear reactors to achieve fusion.

9.2 Electromagnetic Waves in Conductors

As we have mentioned above, the dielectric constant of a material is complex, whether the material is a dielectric or a metal. Whereas for insulators by and large the imaginary part is small and can often be neglected, for metals it can be quite significant or even dominant. We now look at the behaviour of electromagnetic waves in metals in some further detail because of the importance of the subject. The real part of the dielectric constant of a metal is generally called the "dielectric constant" and the imaginary part as the "conductivity". When defined this way, the conductivity, as we have seen, is independent of frequency for microwave and lower frequency region.

According to Ohm's law the (free) current density in a conductor is proportional to the electric field, so that ($\vec{J}$ and σ here refer to the free current and charge densities respectively.)

$$\vec{J} = \sigma\vec{E} \qquad (21)$$

With this, Maxwell's equations in a conductor take the form

$$\text{(i)} \quad \vec{\nabla}.\vec{E} = \rho / \varepsilon \,; \qquad \text{(ii)} \quad \vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \,; \qquad (22)$$

$$\text{(iii)} \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \,; \qquad \text{(iv)} \quad \vec{\nabla}.\vec{B} = 0 \qquad (23)$$

Now from the continuity equation (for free charge)

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (24)$$

together with Ohm's law (21) and Gauss law (i) above, we have

$$\frac{\partial \rho}{\partial t} = -\sigma \vec{\nabla}.\vec{E} = -\frac{\sigma}{\varepsilon}\rho$$

which, on integrating, gives

$$\rho(t) = \rho(0) e^{-\sigma t/\varepsilon}$$

Thus any initial charge density in a conductor dissipates with time, the *characteristic time*, $\tau \equiv \varepsilon/\sigma$. This of course is the well known fact that the charge resides only on the surface but not inside a conductor. The time constant τ is a measure of the goodness of a conductor. For an "ideal" conductor, $\sigma \to \infty; \tau \to 0$ and any charge inside a conductor flows immediately to the surface. A conductor is good as long as τ is much less than other relevant time intervals, say $\tau << 1/\omega$ On the other hand, if $\tau >> 1/\omega$, we have a poor conductor.

After the transient charge disappears, Maxwell's equations inside a conductor take the form

(i) $\vec{\nabla}.\vec{E} = 0$; (ii) $\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} + \mu\sigma\vec{E}$; (22)

(iii) $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$; (iv) $\vec{\nabla}.\vec{B} = 0$ (23)

These equations are the same as in dielectrics except for the last term in equation (ii) of (22). So following the same procedure as we did for the case of vacuum or non-conducting medium (take the curl of equations (ii) and (iii) and use other equations and various vector identities), we obtain

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \qquad (24)$$

$$\nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t} \qquad (25)$$

These equations also admit plane-wave solutions, but this time the wave number k is complex:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{x} - \omega t)}, \qquad \vec{B} = \vec{B}_0 e^{i(\vec{k}.\vec{x} - \omega t)} \qquad (26)$$

$$k = \hat{n}k; \qquad k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \qquad (27)$$

On taking the square root, we write

$$k = k_R + ik_I \qquad (28)$$

where

$$k_R = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+(\frac{\sigma}{\varepsilon\omega})^2}+1\right]^{1/2} \quad (29)$$

$$k_I = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+(\frac{\sigma}{\varepsilon\omega})^2}-1\right]^{1/2} \quad (30)$$

As we can see, for $\sigma = 0$, we recover the results for a dielectric. The imaginary part of wave number leads to an attenuation of the wave;

$$\vec{E} = \vec{E}_0 e^{-k_I\vec{x}.\hat{n}} e^{i(\vec{k}_R.\vec{x}-\omega t)}, \ \vec{B} = \vec{B}_0 e^{-k_I\vec{x}.\hat{n}} e^{i(\vec{k}_R.\vec{x}-\omega t)} \quad (31)$$

The *skin depth d* is defined as

$$d = \frac{1}{k_I} \quad (32)$$

It is the distance at which the amplitude of the wave is reduced to $1/e$ of its original value and is a measure of the distance up to which the wave penetrates into the conductor. The real part of k determines the wavelength, the phase velocity and the refractive index:

$$\lambda = \frac{2\pi}{k_R}; \qquad v = \frac{\omega}{k_R}; \qquad n = \frac{ck}{\omega}. \quad (33)$$

The attenuated plane wave solutions, equations (31) satisfy the modified wave equations (24) and (25). They must, however, also satisfy Maxwell's equations (22) and (23). Substituting the solution into the Maxwell equations we find, as expected, that the waves are transverse and the electric and magnetic fields are transverse to each other, the same as the result obtained for waves in a non-dissipative medium:

$$\vec{E}.\hat{n} = \vec{B}.\hat{n} = \vec{E}.\vec{B} = 0.$$

For definiteness, if we take the direction of propagation to be the z-axis, and the electric field along the x-axis, then the magnetic field is along the y-axis:

$$\begin{aligned}\vec{E}(z,t) &= E_0 e^{-k_I z} e^{i(k_R z-\omega t)}\hat{x}, \\ \vec{B}(z,t) &= B_0 e^{-k_I z} e^{i(k_R z-\omega t)}\hat{y} = \frac{k}{\omega} E_0 e^{-k_I z} e^{i(k_R z-\omega t)}\hat{y}\end{aligned} \quad (34)$$

We can also express the complex wave number, k, in terms of its modulus and phase as

$$k = Ke^{i\phi} \quad (35)$$

Then

$$K = |k| = \sqrt{k_R{}^2 + k_I{}^2} = \omega\sqrt{\varepsilon\mu}\left(1 + (\frac{\sigma}{\varepsilon\omega})^2\right)^{1/4} \tag{36}$$

and

$$\phi = \tan^{-1}(\frac{k_I}{k_R}) = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\varepsilon\omega}) \tag{37}$$

From equation (34) we see that since $B_0 = \frac{k}{\omega}E_0 = \frac{Ke^{i\phi}}{\omega}E_0$, the electric field and magnetic induction are no longer in phase; in fact

$$\delta_{mag} - \delta_{el} = \phi;$$

the magnetic field *lags behind* the electric field by an amount ϕ.

The real amplitudes of $\vec{E}$ and $\vec{B}$ are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\varepsilon\mu}\left(1 + (\frac{\sigma}{\varepsilon\omega})^2\right)^{1/4}.$$

Thus the real electric and magnetic fields are finally

$$\vec{E}(z,t) = E_0 e^{-k_I z}\cos(k_R z - \omega t + \delta_E)\hat{x},$$
$$\vec{B}(z,t) = B_0 e^{-k_I z}\cos(k_R z - \omega t + \delta_E + \phi)\hat{y}$$

[See Figure 9.18 – Griffiths pg 396]

Summary

1. *In this module we describe to the student the propagation of electromagnetic waves in various kinds of media. The velocity in a medium depends on the dielectric constant which in turn depends on the frequency of the propagating wave.*
2. *We first describe propagation in a dielectric medium. For this purpose we employ a simple model of a dielectric and obtain the frequency dependence of the dielectric constant.*
3. *The dielectric constant shows a resonant behaviour as a function of frequency leading to the phenomenon of anomalous dispersion in the intermediate frequency range.*
4. *We then describe to the student the low frequency and high frequency behaviour of the dielectric constant, which are quite different from each other.*
5. *Finally we study the case of metals which are a special case of dielectrics in which the dielectric constant has a very high imaginary part. This leads to a strong attenuation of the wave in a metal which is described in terms of the conductivity of the metal.*